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A SENSITIVITY EQUATION AND ITS USE IN OPTIMAL CONTROL THEORY

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Abstract. The theory of generalized functions is used to obtain a sensitivity equation describing the effect of a change in a parameter on the solution of a differential equation involving piecewise continuous variables. It is shown how this equation may be applied in a parameter search technique for the solution of certain two-point boundary-value problems occurring in optimal control theory. Numerical results are presented for the two-point boundary-value problem occurring in the solution of the fuel optimal lunar rendezvous problem.

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1. Introduction. Consider the scalar differential equation

$$\dot{y} = g(y, \operatorname{sgn} z, \alpha, t),\ t \in [t_o, t_1],\ y(t_o) = y_o \tag{1}$$

with $z = z(y, \alpha, t)$ and

$$\operatorname{sgn} z = \begin{cases} 1 & \text{if } z > 0 \\ -1 & \text{if } z < 0 \\ \text{undefined} & \text{if } z = 0 \end{cases}$$

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The function z is continuous in y, t, and the parameter α. The function g is continuous in y, $\operatorname{sgn} z$, and α and piecewise continuous in t with points of discontinuity at the zeros (assumed

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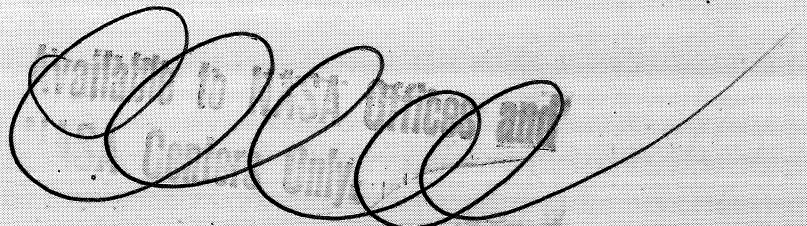
to be finite in number) of $z(y(\alpha, t), \alpha, t)$ for given α. It is a purpose of this paper to derive an equation for $\frac{d}{d\alpha} y(\alpha, t)$ where $y(\alpha, t)$ is the solution for (1) for given α. This derivation is presented in section 2. In section 3 the use of such an equation in the iterative solution of a class of two-point boundary-value problems occurring in optimal control theory is presented. Finally, numerical results are presented in section 4 for the solution of the two-point boundary-value problem resulting from the application of Pontryagin's maximum principle to obtain fuel optimal lunar rendezvous trajectories.

2. Derivation of the equation for $\frac{dy}{d\alpha} (\alpha, t)$. The difficulties in the computation of $\frac{dy}{d\alpha} (\alpha, t)$ due to the possible lack of differentiability of $y(\alpha, t)$ in t at the jump discontinuities of sgn z can be overcome by use of the theory of generalized functions.

The theory of generalized functions can be described as that theory which makes rigorous the use of such familiar symbolic expressions as the Dirac "delta-function". The usual definition of the "δ-function" is a function which vanishes everywhere except at a single point but whose integral over the entire real line is unity. This is not a rigorous mathematical definition since no mathematical function can have these properties.

There are, however, several methods of rigorously making use of such symbolic expressions. In each the concept of a function is generalized to include all continuous functions, all Lebesque integrable functions, and new objects of which a simple example is



the "δ-function". As a result of this generality the theory is referred to as the theory of generalized functions.

Probably the most mathematically sophisticated approach, requiring a knowledge of measure theory and Lebesque integration, is due to L. Schwartz. An introduction to Schwartz's French work [1] has been prepared in English by I. Halperin [2]. Fortunately, there is another formulation, presented in a text by M. T. Lighthill [3] and originally due to G. Temple [4], requiring only a knowledge of ordinary calculus. Lighthill's approach is to construct generalized functions in terms of sequences of continuous functions. For example, a representation (not unique) for the Dirac "δ-function" is the sequence $\left\{e^{-nt^2}(n/\pi)^{1/2}\right\}$. In the limit as $n \to \infty$ $\int_{-\infty}^{\infty} e^{-nt^2}(n/\pi)^{1/2}\,dt \to 1$ and thus $e^{-nt^2}(n/\pi)^{1/2}$ takes on the properties of the "δ-function". In Lighthill's text it is essentially established that sets of equivalent sequences form an additive abelian group over which differentiation is defined. Because of its mathematical simplicity, Lighthill's approach is adopted for use in the computation of $\frac{dy}{d\alpha}(\alpha, t)$. Appendix A contains the needed definitions and properties of Lighthill's formulation.

Begin by assuming that $\frac{dy}{d\alpha}(\alpha, t)$, $\frac{\partial g}{\partial y}\frac{dy}{d\alpha}(\alpha, t)$, and $\frac{\partial g}{\partial \alpha}$ exist as ordinary functions satisfying property 2 of appendix A on $[t_o, t_1]$. Extend y, g, $\frac{dy}{d\alpha}$, and $\frac{\partial g}{\partial \alpha}$ by defining them to be zero outside $[t_o, t_1]$. Also let z be extended outside $[t_o, t_1]$ as any good function so long as $z(y(\alpha, t), \alpha, t) \neq 0$ for $t \notin [t_o, t_1]$. These ordinary functions then exist (by property 2) as generalized functions.

Formally write

$$\frac{dg}{d\alpha} = \frac{\partial g}{\partial \alpha}\frac{dz}{d\alpha}\frac{d(\operatorname{sgn} z)}{dz} + \frac{\partial g}{\partial y}\frac{dy}{d\alpha} + \frac{\partial g}{\partial \alpha}\,.$$

Let t^* be a zero of $z(y(\alpha, t), \alpha, t)$ and $S(t^*)$ be the set of all zeros on $[t_o, t_1]$. If $\frac{d}{dt} z(y(\alpha, t), \alpha, t)\Big|_{t=t^*} \neq 0$ for given $y(\alpha, t)$ and α then by Riemann-Stieltjes integration

$$\int_{t_o}^{t_1} \frac{\partial g}{\partial \alpha}\frac{dz}{d\alpha}\frac{d \operatorname{sgn} z}{d\alpha}\,dt = \sum_{S(t^*)} Q(t^*)$$

where

$$Q(t^*) = -2 \operatorname{sgn} z(t^{*-}) \frac{\partial g(t^*)}{\partial(\operatorname{sgn} z)} \frac{\frac{dz}{d\alpha}(t^*)}{\frac{dz}{dt}(t^*)}$$

and $\sum_{S(t^*)}$ represents the sum over all t^* in $[t_o, t_1]$. Thus a represention for $\frac{\partial g}{\partial(\operatorname{sgn} z)}\frac{dz}{d\alpha}\frac{d(\operatorname{sgn} z)}{dz}$ as a generalized function is $\sum_{S(t^*)} Q(t^*)\,\delta(t - t^*)$ where $\delta(t - t^*)$ is the Dirac "δ-function" of $t - t^*$. By property 3 of the appendix the generalized function equation

$$\frac{d}{dt}\left(\frac{dy}{d\alpha}\right) = \frac{d}{d\alpha}(\dot{y}) = \sum_{S(t^*)} Q(t^*)\,\delta(t - t^*) + P(t)$$

can be written where $P(t)$ is the generalized function corresponding to the ordinary function $\frac{\partial g}{\partial y}\frac{dy}{d\alpha} + \frac{\partial g}{\partial \alpha}$.

Applying first property 1 and then 2 of the appendix the ordinary function equation (2) results for all "good" functions $F(t)$.

$$\int_{t_o}^{t_1} \left[\frac{dy}{d\alpha}(\alpha, t) - \frac{dy}{d\alpha}(\alpha, t_o) - \sum_{S(t^*)} Q(t^*)\, H(t - t^*) \right.$$

(2)

$$\left. - \int_{t_o}^{t} \left(\frac{\partial g}{\partial y} \frac{dy}{d\alpha} + \frac{\partial g}{\partial \alpha} \right) ds \right] \frac{dF(t)}{dt}\, dt = 0$$

with

$$H(t - t^*) = \begin{cases} 0 & t < t^* \\ 1 & t > t^* \\ \text{undefined} & t = t^* \end{cases}$$

Equation (2) leads by use of the integral form of the fundamental lemma of the calculas of variations [5] to the integral equation

$$(3) \quad \frac{dy}{d\alpha}(\alpha, t) = \frac{dy}{d\alpha}(\alpha, t_o) - 2 \sum_{S(t^*)} \operatorname{sgn} z(t^{*-}) \frac{\partial g(t^*)}{\partial(\operatorname{sgn} z)} \frac{\frac{dz}{d\alpha}(t^*)}{\frac{dz}{dt}(t^*)}$$

$$H(t - t^*) + \int_{t_o}^{t} \left(\frac{\partial g(s)}{\partial y} \frac{dy(\alpha, s)}{d\alpha} + \frac{\partial g(s)}{\partial \alpha} \right) ds .$$

The initial value $\frac{dy}{d\alpha}(\alpha, t_o)$ is determined by the nature of α and its relation to $y(\alpha, t)$. Given the solution $y(\alpha, t)$ of (1), equation (3) determines $\frac{dy}{d\alpha}(\alpha, t)$ on $[t_o, t_1]$.

3. A use in optimal control theory. The general result of applying an optimal control theory such as Pontryagin's maximum principle [6] to a controllable dynamic system is a two-point boundary-value problem. A large class of these boundary-value problems take the form:

I. Given the matrix equation $\dot{\bar{y}} = \bar{g}(\bar{y}, \operatorname{sgn} \bar{z}, \bar{\alpha}, t)$, $t \in [t_o, t_1]$, $\bar{y}(\bar{\alpha}, t_o) = \bar{y}_o$, determine the parameter vector $\bar{\alpha}$ such that $\bar{e}(\bar{y}(\alpha, t_1), \alpha, t_1) = 0$.

The variable $\bar{y}(\bar{\alpha}, t)$ is a vector made up of the elements of the state and conjugate variables of the control theory, $\bar{\alpha}$ is a system parameter usually some of the elements of $\bar{y}(\bar{\alpha}, t_o)$ or t_1, and $\bar{e}$ represents terminal conditions. The variable $\bar{z}(\bar{y}, (\bar{\alpha}, t), \alpha, t)$ is a vector of continuous switching functions. It is assumed that the vector quantities of I satisfy the same requirements as imposed on the corresponding scalars in section 2.

There are a variety of search methods available for the solution of nonlinear algebraic equations or algebraic equations involving unknown parameters. For example there are the first-order methods of gradients, conjugate gradients, Davidon, and Newton Raphson, not to mention higher order approaches. A review of these methods is found in [7].

If the terminal condition $\bar{e}(\bar{y}(\bar{\alpha}, t_1), \bar{\alpha}, t_1) = \bar{0}$ is viewed as a nonlinear algebraic equation in $\bar{\alpha}$ and one of the first-order methods applied then it will be necessary to compute

$$\frac{d\bar{e}}{d\bar{\alpha}} \left(y(\bar{\alpha}, t_1), \bar{\alpha}, t_1\right).$$

This matrix is given by

$$\frac{d\bar{e}}{d\bar{\alpha}} \left(\bar{y}(\bar{\alpha}, t_1), \alpha, t_1\right) = \frac{\partial \bar{e}}{\partial \bar{y}} \left(\bar{y}(\bar{\alpha}, t_1), \bar{\alpha}, t_1\right) \frac{dy(\bar{\alpha}, t_1)}{d\bar{\alpha}} + \frac{\partial \bar{e}}{\partial \bar{\alpha}} \left(y(\bar{\alpha}, t_1), \alpha, t_1\right) \tag{4}$$

The vector $\bar{y}(\bar{\alpha}, t_1)$ is given by integration of

$$\dot{\bar{y}} = \bar{g}(\bar{y}, \operatorname{sgn} \bar{z}, \bar{\alpha}, t) \tag{5}$$

to t_1 for given $z(\bar{y}(\bar{\alpha}, t), \bar{\alpha}, t)$ and $\bar{\alpha}$ and the matrix $\frac{d\bar{y}}{d\bar{\alpha}}(\bar{\alpha}, t_1)$ easily follows from the discussion in section 2.

Let the elements of $\bar{z}(\bar{y}(\bar{\alpha}, t), \alpha, t)$ be $z_q(\bar{y}(\bar{\alpha}, t), \alpha, t)$, $q = 1, 2, \cdots, l$ and $S_q(t^*)$ be the set of zeros of z_q on $[t_o, t_1]$. Also let the elements of $\bar{y}$ be y_i, $i = 1, 2, \cdots, n$, $\bar{g}$ be g_i, $i = 1, 2, \cdots, n$ and $\bar{\alpha}$ be α_i, $i = 1, 2, \cdots, m$. By observation of (3) the elements of $\frac{d\bar{y}}{d\bar{\alpha}}(\bar{\alpha}, t_1)$, $\frac{dy_i}{d\alpha_j}(\bar{\alpha}, t_1)$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$, are given by the simultaneous solution of the integral equations

$$\begin{aligned}\frac{dy_i}{d\alpha_j}(\bar{\alpha}, t) = \frac{dy_i}{d\alpha_j}(\bar{\alpha}, t_o) + \sum_{q=1}^{l} \sum_{S_q(t^*)} & - 2 \operatorname{sgn} z_q(t^{*-}) \frac{\partial g_i}{\partial \operatorname{sgn} z_q}(t^*) \frac{\frac{dz_q}{d\alpha_j}(t^*)}{\frac{dz_q}{dt}(t^*)} H(t - t^*) \\ & + \int_{t_o}^{t} \left[\sum_{k=1}^{n} \left(\frac{\partial g_i}{\partial y_k} \frac{dy_k}{d\alpha_j}(\bar{\alpha}, s) + \frac{\partial g_i}{\partial \alpha}(s) \right) \right] ds\end{aligned} \tag{6}$$

to t_1 for given $\bar{y}(\bar{\alpha}, t)$ and $\bar{\alpha}$. Thus through equations (4), (5), and (6) the two-point boundary-value problem I may be attacked by algebraic search techniques.

4. An application. By way of application the following problem was considered:

Find the optimal bounded thrust magnitude and direction of a space vehicle, launched from the surface of the moon, such that it will rendezvous in a given amount of time with a minimum amount of fuel expenditure with a target vehicle in a circular orbit about the moon. The moon is assumed to have an inverse square law gravitational field. Rendezvous is interpreted to be the nulling of the relative position and velocity of the two vehicles.

Pontryagin's maximum principle was applied to the three-dimensional equations describing the relative motion of the two spacecraft yielding an optimal thrust magnitude and direction and a two-point boundary-value problem of the form I. In this case the vector $\bar{y}$ had as elements the relative position and velocity and the additional conjugate variables introduced by the maximum principle. That is, with reference to Figure 1, $\bar{y}(\bar{\alpha}, t) = \text{col.} \left(y_i (\bar{\alpha}, t)\right)$, $i = 1, 2, \cdots, 12$ where

$$y_1 (\bar{\alpha}, t) = \hat{i} \cdot \left[\bar{R}_v (\bar{\alpha}, t) - \bar{R}_S (t)\right]$$

$$y_3 (\bar{\alpha}, t) = \hat{j} \cdot \left[\bar{R}_v (\bar{\alpha}, t) - \bar{R}_S (t)\right]$$

$$y_5 (\bar{\alpha}, t) = \hat{k} \cdot \left[\bar{R}_v (\bar{\alpha}, t) - \bar{R}_S (t)\right]$$

$$y_{i+1} (\bar{\alpha}, t) = \frac{d}{dt} y_i (\bar{\alpha}, t), \quad i = 1, 3, 5$$

$y_{i+6} (\bar{\alpha}, t)$ = the conjugate variable corresponding to $y_i (\bar{\alpha}, t)$, $i = 1, 2, \cdots, 6$

and

$\hat{i}, \hat{j}, \hat{k}$ = orthogonal unit vectors

$\bar{R}_S(t)$ = instantaneous position vector of target vehicle

$\bar{R}_V(\bar{\alpha}, t)$ = instantaneous position vector of the intercepter vehicle.

Figure 1.- Geometry of the lunar rendezvous problem.

The elements of the parameter vector $\bar{\alpha}$ were the initial values of the conjugate variables. That is

$$\bar{\alpha} = \text{col.}\left(\alpha_i\right)$$

$$\alpha_i = y_{i+6}\left(t_o\right) \quad i = 1, 2, \cdots, 6 \ .$$

Also the terminal error vector was made up of the relative positions and velocities at the rendezvous time, t_1, or

$$\bar{e}\left(\bar{\alpha}, t_1\right) = \text{col.}\left(e_i\left(\bar{\alpha}, t_1\right)\right)$$

$$e_i\left(\bar{\alpha}, t_1\right) = y_i\left(\bar{\alpha}, t_1\right), \; i = 1, 2, \cdots, 6 \; .$$

There was a single switching function, $z(\bar{y}(\bar{\alpha}, t), \bar{\alpha}, t)$, such that the target vehicle enters a coasting period when $z < 0$ and burns with maximum thrust when $z > 0$. Transition points occur where $z(\bar{y}, \bar{\alpha}, t), \bar{\alpha}, t) = 0$.

A parameter search technique similar to the one presented by Marquardt [8] was adopted to solve this boundary-value problem. Complete details are presented in [9]. Basically the method takes an assumed value of $\bar{\alpha}$ for which the scalar

$$E\left(\bar{y}\left(\bar{\alpha}, t_1\right), \bar{\alpha}, t_1\right) = \bar{e}\left(\bar{y}\left(\bar{\alpha}, t_1\right), \bar{\alpha}, t_1\right) \cdot B \, \bar{e}\left(\bar{y}\left(\bar{\alpha}, t_1\right), \bar{\alpha}, t_1\right) \neq 0$$

and corrects $\bar{\alpha}$ by

$$\delta\bar{\alpha} = -\left[\frac{d\bar{e}'}{d\bar{\alpha}}\left(\bar{y}\left(\alpha, t_1\right), \bar{\alpha}, t_1\right) B \frac{d\bar{e}}{d\bar{\alpha}}\left(\bar{y}\left(\bar{\alpha}, t_1\right), \bar{\alpha}, t_1\right) + \lambda I\right]^{-1}$$

$$\frac{d\bar{e}'}{d\bar{\alpha}}\left(\bar{y}\left(\bar{\alpha}, t_1\right), \bar{\alpha}, t_1\right) B \, \bar{e}\left(\bar{y}\left(\bar{\alpha}, t_1\right), \bar{\alpha}, t_1\right)$$

where B is a positive definite diagonal weighing matrix and the prime (') denotes matrix transpose. For an appropriately chosen $\lambda > 0$

$$E\left(\bar{y}\left(\bar{\alpha} + \delta\bar{\alpha}, t_1\right) \bar{\alpha} + \delta\bar{\alpha}, t_1\right) < E\left(\bar{y}\left(\bar{\alpha}, t_1\right), \bar{\alpha}, t_1\right) .$$

The matrix $\frac{d\bar{e}}{d\bar{\alpha}}\left(\bar{y}\left(\bar{\alpha}, t_1\right), \bar{\alpha}, t_1\right)$ is found by equations corresponding to (4), (5), and (6).

Successive application of the foregoing process from an assumed value of $\bar{\alpha}$ such that $E\left(\bar{y}\left(\bar{\alpha}, t_1\right), \bar{\alpha}, t_1\right) \neq 0$ leads to a monotone decreasing sequence of $E\left(\bar{y}\left(\bar{\alpha}, t_1\right), \bar{\alpha}, t_1\right)$. The algorithm converged well yielding both planar and nonplanar minimum fuel optimal trajectories with equal ease. A typical example of the convergence properties is shown by Table 1. Table 1 depicts an inplane $t_1 = 1300$ seconds rendezvous solution using the results of a previously obtained 1000-second solution as a nominal. Since the solution is planar $\alpha_5 = \alpha_6 = 0$. At $t_o = 0$ seconds the interceptor vehicle left the surface of the moon when, as seen in Figure 2, the space station was 13.7^o behind and in a 100-nautical-mile circular orbit about it.

Figure 2.- Launch conditions corresponding to Table 1.

The matrix $B = \text{diag}\left(b_i\right)$, $b_i = 1$, $i = 1, 3, 5$, $b_i = 10$, $i = 2, 4, 6$, and $\lambda = 10$. At 1300 seconds rendezvous occurred after one coast period with 53.5 percent of the original launch weight remaining.

A complete discussion of other planar and nonplanar minimum fuel results, as well as time optimal solutions, obtained by letting t_1 be an element of $\bar{\alpha}$, is found in [9].

Table 1

A fuel optimal iteration sequence, t_1 = 1300 seconds

Iteration	First coast time, sec	Second burn time, sec	Unknown parameters				Error criterion $E(\bar{y}(\bar{\alpha}, t_1), \bar{\alpha}, t_1)$
			α_1	α_2	α_3	α_4	
Nominal[a]							
t_1 = 1000 sec	334.6	959.2	0.37087	2766.5	4.6379	1477.5	0.101×10^{13}
1	360.8	1296.1	.40192	2710.0	3.9214	1091.3	$.129 \times 10^{12}$
2	343.9	1268.7	.29196	2734.2	4.0687	1256.0	$.776 \times 10^{9}$
3	342.3	1259.2	.14509	2663.0	3.9937	1244.8	$.306 \times 10^{8}$
4	342.6	1262.5	.036295	2611.7	3.9098	1221.7	$.228 \times 10^{8}$
5	342.5	1264.7	- .035214	2577.6	3.8512	1206.4	$.726 \times 10^{7}$
6	342.5	1266.5	- .081362	2555.7	3.8110	1196.1	$.234 \times 10^{7}$
7	342.5	1267.7	- .11073	2541.7	3.7844	1189.3	$.596 \times 10^{6}$
8	342.5	1268.4	- .12931	2532.8	3.7672	1184.9	$.137 \times 10^{6}$
9	342.5	1268.8	- .14105	2527.2	3.7561	1182.1	$.290 \times 10^{5}$
10	342.5	1269.0	- .14846	2523.7	3.7491	1180.4	$.563 \times 10^{4}$
11	342.5	1269.1	- .15314	2521.5	3.7446	1179.2	$.105 \times 10^{4}$
12	342.5	1269.1	- .15610	2520.0	3.7418	1178.5	204.
13	342.5	1269.1	- .15794	2519.2	3.7400	1178.0	33.8
14	342.5	1269.1	- .15910	2518.6	3.7389	1177.8	12.0
15	342.5	1269.1	- .15984	2518.3	3.7382	1177.6	1.67
16[b]	342.5	1269.1	- .16030	2518.1	3.7377	1177.5	.600

[a] $y_1(\bar{\alpha}, t_1) = 0.7616 \times 10^{6}$ ft

$y_2(\bar{\alpha}, t_1) = 0.5222 \times 10^{4}$ ft/sec

$y_3(\bar{\alpha}, t_1) = -0.1232 \times 10^{7}$ ft

$y_4(\bar{\alpha}, t_1) = -0.1056 \times 10^{5}$ ft/sec

[b] $y_1(\bar{\alpha}, t_1) = -0.1824$ ft

$y_2(\bar{\alpha}, t_1) = -0.2343$ ft/sec

$y_3(\bar{\alpha}, t_1) = -0.3694$ ft

$y_4(\bar{\alpha}, t_1) = -0.2192$ ft/sec

Appendix A: Generalized Functions. Certain definitions and properties of generalized functions as presented in reference 3 are contained in this section for reference in the main text and to give the reader a general idea of the concepts involved.

Definition 1 - A good function, $F(t)$, is one which is everywhere differentiable any number of times such that it and all its derivatives approach zero as its argument approaches $\pm\infty$.

Definition 2 - A sequence $h_n(t)$ of good functions is called regular if, for any good function $F(t)$, the limit

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} h_n(t)\, F(t) \quad \text{exists.}$$

Definition 3 - A generalized function, $h(t)$, is defined as a regular sequence $h_n(t)$ of good functions, that is, $h(t) \to [h_n(t)]$.

Thus each generalized function is really the class of all regular sequences equivalent, in the sense that the limit in Definition 2 is the same for each sequence, to a given regular sequence.

By the symbol $\int_{-\infty}^{\infty} h(t)\, F(t)\, dt$ one means $\lim_{n \to \infty} \int_{-\infty}^{\infty} h_n(t)\, F(t)\, dt$.

Definition 4 - Two generalized functions, $h(t)$ and $g(t)$, are equal if and only if

$$\int_{-\infty}^{\infty} h(t)\, F(t)\, dt = \int_{-\infty}^{\infty} g(t)\, F(t)\, dt$$

for all good functions $F(t)$.

Definition 5 - The derivative $h'(t)$ is defined by the sequence $h'_n(t)$, that is, $h'(t) \to [h_n'(t)]$.

Property 1 - $\int_{-\infty}^{\infty} h'(t)\, F(t)\, dt = -\int_{-\infty}^{\infty} h(t)\, F'(t)\, dt.$

Property 2 - If the ordinary function $f(t)$ is such that $\int_{-\infty}^{\infty} (t^2 + 1)^{-n} f(t)\, dt$ exists (for some natural number N) then a generalized function $h(t)$ exists such that for all $F(t)$

$$\int_{-\infty}^{\infty} h(t)\, F(t)\, dt = \int_{-\infty}^{\infty} f(t)\, F(t)\, dt.$$

The integral on the right is the integral in the ordinary sense. When the generalized function $h(t)$ has been defined, this integral has a meaning also in the theory of generalized functions and the above equation states that these two meanings are the same.

If

$$\operatorname{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$ Signum function

$$H(t) = 1/2\,(1 + \operatorname{sgn} t)$$ Heaviside function

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \text{at } t = 0,\ \delta(t) \text{ is infinite in such a way that} \\ \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \end{cases}$$ Dirac "δ-function"

then $\frac{d}{dt} \operatorname{sgn} t = 2 \frac{dH}{dt} = 2\delta(t).$

Definition 6 - If $h_\alpha(t)$ is a generalized function of t for each value of a parameter α then

$$\frac{d}{d\alpha} h_\alpha(t) = \lim_{\alpha' \to \alpha} \frac{h_{\alpha'}(t) - h_\alpha(t)}{\alpha' - \alpha}$$

Property 3 - If $\frac{dh_\alpha(t)}{d\alpha}$ exists, $\frac{d}{d\alpha}\frac{d}{dt}h_\alpha(t) = \frac{d}{dt}\frac{d}{d\alpha}h_\alpha(t)$.

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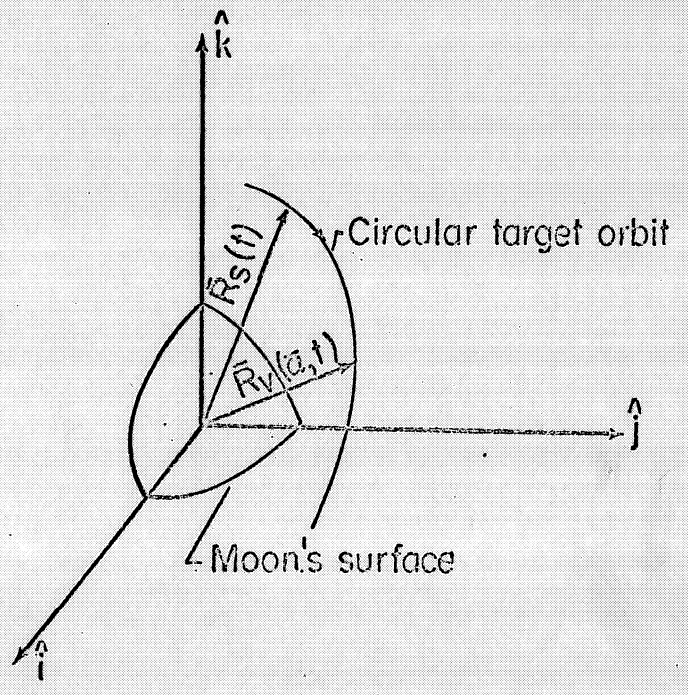
k̂
Circular target orbit
$\bar{R}_S(t)$
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Moon's surface
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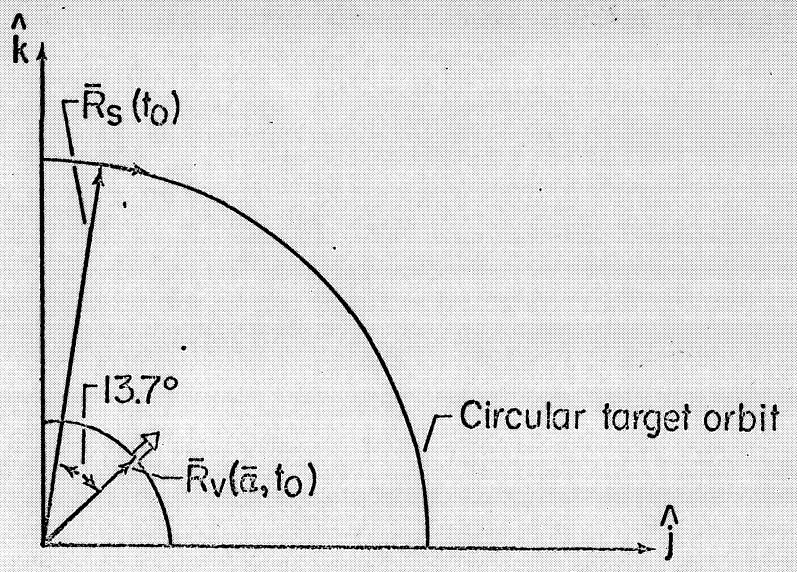

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